STANDING SOLITARY EULER-KORTEWEG WAVES ARE UNSTABLE

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1. The result

The Euler-Korteweg system is given by the equations

$$(1)\qquad \begin{aligned} V_t - U_y &= 0, \\ U_t + p(V)_y &= -(\kappa(V)V_{yy} + \frac{1}{2}(\kappa(V))_y V_y)_y, \end{aligned}$$

with $\kappa(V) > 0$. System (1), and notably its solitary waves

$$\begin{pmatrix} V \\ U \end{pmatrix}(x,t) = \begin{pmatrix} v \\ u \end{pmatrix}(x - ct) \quad \text{with} \quad \begin{pmatrix} v \\ u \end{pmatrix}(\pm\infty) = \begin{pmatrix} v_* \\ u_* \end{pmatrix},$$

have been intensely studied by Benzoni-Gavage et al. in [1, 2, 3].

Definition 1. [3] *A traveling wave (v,u) of (1) is called orbitally stable if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for any solution $(V,U) \in (v,u) + C([0,T); H^3(\mathbb{R}) \times H^2(\mathbb{R}))$ of (1), closeness at initial time,*

$$\|(V,U)(\cdot,0) - (v,u)(\cdot)\|_{H^1 \times L^2} < \delta$$

implies closeness at any time

$$\inf_{\sigma \in \mathbb{R}} \|(V,U)(\cdot,t) - (v,u)(\cdot + \sigma)\|_{H^1 \times L^2} < \varepsilon \quad \text{for all } t > 0.$$

The following is the point of this short note.

Theorem 1. *All non-trivial standing solitary Euler-Korteweg waves are not orbitally stable.*

2. The proof

For fixed base state v_*, the solitary waves homoclinic to v_* occur in families (u^c, v^c) parametrized by their speed c. The proof of Theorem 1 is based on

Lemma 1. [1] *A solitary wave (u^{c_*}, v^{c_*}) is orbitally unstable if the moment of instability*

$$m(c) = \int_{-\infty}^{\infty} \kappa(v) v'^2 \, d\xi$$

is not convex at $c = c_*$.

Theorem 1 follows from

Lemma 2. $m''(0) < 0$.

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To prove Lemma 2, we recall that with

$$F(v,c) = -f(v) + f(v_*) - p(v_*)(v - v_*) + \frac{1}{2}c^2(v - v_*)^2, \quad -\frac{df(v)}{dv} = p(v), \tag{2}$$

the profile equation

$$\kappa(v)v'' + \frac{1}{2}(\kappa(v))'v' = -\frac{\partial F(v,c)}{\partial v}$$

possesses (cf. [3]) a first integral given by

$$I(v,v') = \frac{1}{2}\kappa(v)v'^2 + F(v,c). \tag{3}$$

As

$$m(c) = 2\int_{v_*}^{v_m(c)} \kappa(v)v' \, dv$$

with $v_*, v_m(c) > v_*$ consecutive zeros of $F(\cdot,c)$. Since $I(v,v') \equiv 0$ along solutions, we have

$$\begin{aligned} m(c) &= 2\int_{v_*}^{v_m(c)} (\kappa(v))^{1/2} \, (-2F(v,c))^{1/2} \, dv \\ &= 4\int_0^{(v_m(c)-v_*)^{1/2}} \left(\kappa(v_m(c) - w^2)\right)^{1/2} \left(-2F(v_m(c) - w^2, c)\right)^{1/2} \, w \, dw, \end{aligned}$$

where $w := (v_m(c) - v)^{1/2}$ (cf. [6]). The first derivative of m is

$$\begin{aligned} m'(c) &= 4\int \frac{d}{dc}\left\{\left(\kappa(v_m(c) - w^2)\right)^{1/2} \left(-2F(v_m(c) - w^2, c)\right)^{1/2}\right\} w \, dw \\ &= 4\int \frac{\kappa_v(v_m(c) - w^2)v_m'(c)}{2\left(\kappa(v_m(c) - w^2)\right)^{1/2}} \left(-2F(v_m(c) - w^2, c)\right)^{1/2} \, w \, dw \; + \\ &\quad + 4\int \frac{\left(\kappa(v_m(c) - w^2)\right)^{1/2} \left(-F_v(v_m(c) - w^2, c)v_m'(c) - F_c(v_m(c) - w^2, c)\right)}{\left(-2F(v_m(c) - w^2, c)\right)^{1/2}} \, w \, dw \end{aligned}$$

which, due to

$$\frac{\partial}{\partial v}\left((\kappa(v)(-2F(v,c)))^{1/2}\right) = \frac{\kappa_v(v)(-2F(v,c))^{1/2}}{2(\kappa(v))^{1/2}} - \frac{\kappa(v)^{1/2}F_v(v,c)}{(-2F(v,c))^{1/2}},$$

simplifies to

$$m'(c) = -4\int \frac{\left(\kappa(v_m(c) - w^2)\right)^{1/2} F_c(v_m(c) - w^2, c)}{\left(-2F(v_m(c) - w^2, c)\right)^{1/2}} \, w \, dw.$$

The second derivative of m can be written in the form

$$m''(c) = 2\int_{v_*}^{v_m(c)} \frac{A(v,c) + B(v,c)}{(\kappa(v))^{1/2} \, (-2F(v,c))^{3/2}} \, dv$$

with $A(v,c) = F(v,c)\kappa_v(v)v_m'(c)F_c(v,c)$ and
$B(v,c) = \kappa(v)\left((v - v_*)\left(2F(v,c)\left((v - v_*) + 2cv_m'(c)\right) - c(v - v_*)\left(F_v(v,c)v_m'(c) + F_c(v,c)\right)\right)\right).$
As $F_c(v,0) = 0 = A(v,0)$ and

$$B(v,0) = 2\kappa(v)(v - v_*)^2F(v,0) < 0 \quad \text{for all } v \in (v_*, v_m(0)),$$

we indeed have $m''(0) < 0$.

Remark. Certain of the results by Zumbrun [7] on the Bona-Sachs model ($\kappa(V) \equiv 1$ and $p(V) = -V + V^q$ with $q \geq 2$) and by De Bouard [4] on a Gross-Pitaevskii model ($\kappa(V) = 1/(4V^4)$ and $p(V) = \alpha/V^2 - \beta/V^3$ with $\alpha, \beta > 0$) appear as interesting special cases of Theorem 1.

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